

Out-of-equilibrium Correlated Systems : Bipartite Entanglement as a Probe of Thermalization

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Abstract. Thermalization play a central role in out-of-equilibrium physics of ultracold atoms or electronic transport phenomena. On the other hand, entanglement concepts have proven to be extremely useful to investigate quantum phases of matter. Here, it is argued that *bipartite* entanglement measures provide key information on out-of-equilibrium states and might therefore offer stringent thermalization criteria. This is illustrated by considering a global quench in an (extended) XXZ spin-1/2 chain across its (zero-temperature) quantum critical point. A non-local *bipartition* of the chain *preserving translation symmetry* is proposed. The time-evolution after the quench of the *reduced* density matrix of the half-system is computed and its associated (time-dependent) entanglement spectrum is analyzed. Generically, the corresponding entanglement entropy quickly reaches a "plateau" after a short transient regime. However, in the case of the integrable XXZ chain, the low-energy entanglement spectrum still reveals strong time-fluctuations. In addition, its infinite-time average shows strong deviations from the spectrum of a Boltzmann thermal density matrix. In contrast, when the integrability of the model is broken (by small next-nearest neighbor couplings), the entanglement spectra of the time-average and thermal density matrices become remarkably similar.

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1. Introduction

Rapid progress in the field of ultracold atoms [1] offer breed new perspectives to realize controlled experimental setup to investigate out-of-equilibrium physics. Real-time observation of quantum dynamics of isolated systems have become possible [2]. In addition, ultracold atoms loaded on optical lattices [3, 1] or laser-cooled Coulomb crystals of charged ions [4] offer very clean experimental implementation of simple lattice many-body Hamiltonians and provide simulators for Condensed Matter. The ability to dynamically change parameters [5] in these Hamiltonians on short time scales could be exploited to realize quantum information processing [1] or cooling [6] devices.

In electronic condensed matter systems, relaxation towards steady states play a central role in many transport phenomena, like e.g. in electric transport resulting from the application of a sudden voltage bias at the edges of a quantum dot [7] or of a Hubbard chain [8]. Spin chains also offers simple generic systems to investigate out-of-equilibrium physics as e.g. heat transport [9]. However, conceptually, thermalization [10] of non-equilibrium *isolated* quantum many-body systems after e.g. a sudden change of Hamiltonian parameters (quantum quench) is still poorly understood, despite recent work on correlated bosons [11, 12] in one-dimension (1D). Generally, whether some local observables approach steady values and whether their time average equal the corresponding thermal average are often used as criteria of thermalization. However, the fact that thermalization occur for certain local observables (according to the above criteria) does not at all guarantee that other observables will also meet the criteria.

Independently, quantum information concepts have been applied with great success to several domains of Condensed Matter [13], giving new type of physical insights on exotic quantum phases. Quantum entanglement of a A/B bipartition of a many-body (isolated) quantum system can be characterized by the *groundstate* (GS) reduced density matrix (RDM) ρ_A obtained from tracing out the B part. The corresponding entanglement Von Neumann (VN) entropy $S_{\text{VN}} = -\text{Tr}\{\rho_A \ln \rho_A\}$ offers an extraordinary tool [14], e.g. to identify underlying conformal field theory (CFT) structure in one-dimensional systems. Another central quantity is the *entanglement spectrum* (ES) defined by the (positive) eigenvalues of a (dimensionless) pseudo-Hamiltonian \mathcal{H} defined by the implicit relation $\rho_A = \exp(-\mathcal{H})$. Remarkably, ES faithfully reflect CFT structures [15], topological symmetries [16] or properties of *edge states* in fractional quantum Hall states [17] or low-dimensional quantum magnets [18].

So far, time evolution of entanglement has been investigated only in very simple cases e.g. for a small segment in a 1D system after global or local quenches [20, 13, 14]. However, the full potential of entanglement measures has not been fully exploited to investigate thermalization of many-body systems. Because for a given *bipartition* (i.e. into two halves) of the whole system all the information of the GS is contained in its Schmitt decomposition and the ES is a (convenient) way of arranging the Schmitt coefficients, the ES contains the whole information of the state. The main goal here is therefore to use the bipartite ES to take the place of local observables to investigate

thermalization: whether the ES approach the spectrum of a thermal density matrix (whose effective temperature is fixed by the excess energy brought in the quench) is therefore proposed as a stringent test of thermalization. If the ES satisfies this thermalization criterium, other observables would also do. The proposed choice of a bi-partition of the total system is crucial since (i) an extensive subsystem is considered so that thermalization of local and *non-local* observables is addressed at once, (ii) finite size scaling can be performed and (iii) reduced density matrices benefit from numerous conserved quantities (total momenta and total spin) allowing for a direct comparison of the ES *separately in each symmetry sector* and hence an ultimate comparison between time-averaged and Boltzman thermal density matrices. Note that non-local real-space partitions have also been used e.g. to define non-local order in *gapless* spin chains [19].

Practically, a genuine correlated anisotropic 1D spin chain and the time evolution of its reduced density matrix, entanglement entropy and ES after a global quench are considered using exact (Lanczos and full) diagonalization techniques. For the integrable case we have considered, the reduced density matrix exhibits strong time-fluctuations and its infinite-time average significantly deviates from a thermal density matrix, despite the fact that the entanglement entropy reaches a well-defined entropy plateau. In contrast, thermalization, as defined by the above criteria, seems to be possible when integrability is broken by adding (small) extra terms to the Hamiltonian.

2. Model and setup

Let us now consider the 1D anisotropic spin-1/2 Heisenberg (so-called XXZ) model (Fig. 1),

$$H = J_z \sum_i S_i^z S_{i+1}^z + \frac{1}{2} J_{xy} (S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+), \quad (1)$$

whose (1D) parameter space can be mapped on a (half) unit circle assuming $J_{xy} = \cos \theta$ and $J_z = \sin \theta$. Alternatively the system can be viewed as a (half-filled) hard-core boson chain with hopping $t = J_{xy}/2$ and nearest neighbor (NN) repulsion $V = J_z$. Its remarkable phase diagram obtained by Haldane [21] and shown in Fig. 1(b) exhibits a Quantum Critical Point (QCP) located exactly at the SU(2)-symmetric point $\theta = \pi/4$ separating a gapped Charge Density Wave (CDW) insulating phase (Ising phase in the spin language) and a gapless metallic Luttinger Liquid (XY phase). Phase Separation (PS) occurring for an attractive interaction $|V| > 2t$ ($\theta < -\pi/4$) will not be of interest here. Ultimately, we shall consider adding next NN hopping t' and repulsion V' (see Fig. 1(a)) in order to break integrability.

The key to construct extensive quantities is to realize a non-local partition of the chain into "even" and "odd" sites. In other words, if the chain is drawn in a zig-zag fashion as in Fig. 1(a), the A and B parts form the two edges of the system which become explicit for $t' \neq 0$ and $V' \neq 0$. One consider here finite chains of N ($= 16, 20$ and 24) sites with periodic boundary conditions, as shown in Fig. 2. The *groundstate* RDM of the subsystems $\rho_A = \rho_B$ can be computed using translation symmetry of each $L = N/2$

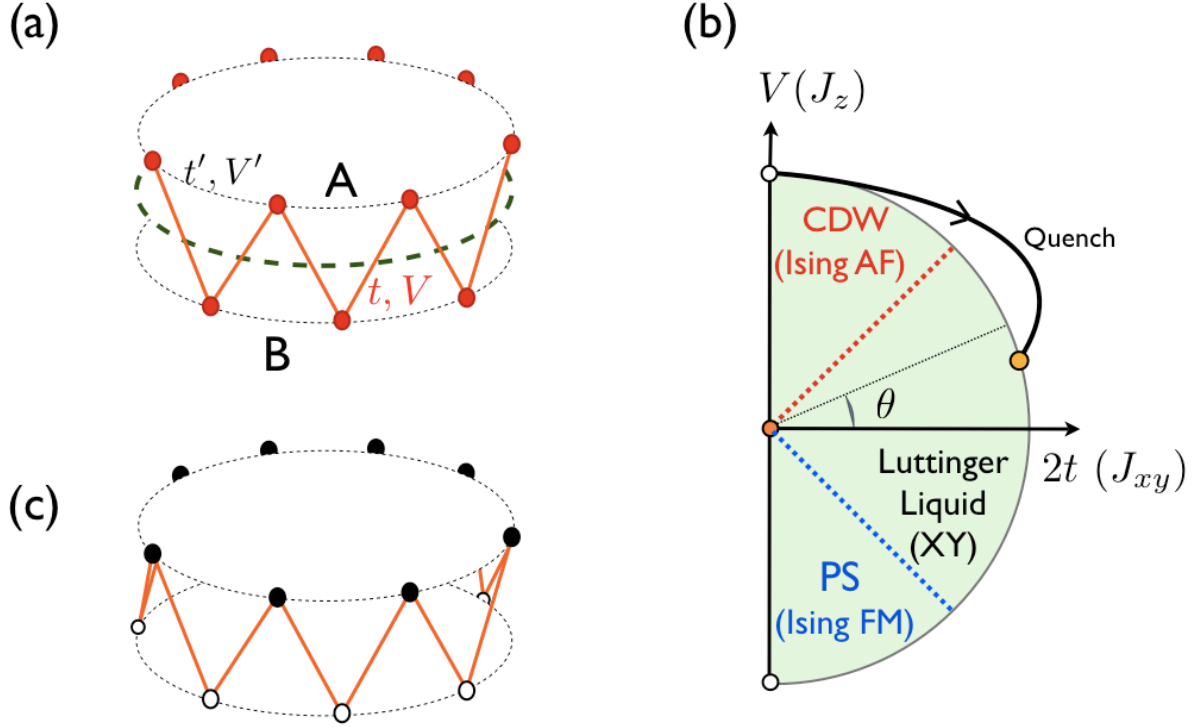


Figure 1. (Color online) (a) A N -site (extended) XXZ spin-1/2 chain (drawn here as a "zig-zag" on a periodic ribbon) is partitioned into two identical A and B subsystems of $L = N/2$ sites by cutting bonds along the dashed line. (b) Phase diagram of the XXZ spin (or hardcore boson) chain mapped onto a circle assuming $J_{xy} = \cos \theta$ and $J_z = \sin \theta$. (c) The CDW groundstate for $\theta = \pi/2$ i.e. $t = 0$ ($J_{xy} = 0$) prepared before the quench : all bosons (up spins) are located on A and the B sites are empty (down spins).

site subsystem [18] (each symmetry class is labelled by a momentum $K = 2\pi n/L$) and the conservation of the Z-component of the total spin (i.e. the number of bosons), $S_A^Z + S_B^Z = 0$. Interestingly, liquid and insulating bosonic phases are characterized by qualitatively different entanglement properties. First, the entanglement entropy *per unit length*, shown in Fig. 2(a) for a $N = 24$ site chain, shows a (kink-like) maximum at the SU(2)-symmetric QCP and vanishes (in the thermodynamic limit) for the classical CDW (Ising) configuration obtained when $\theta \rightarrow \pi/2$. Indeed, as shown in Fig. 1(c), for $\theta = \pi/2$ the (two-fold degenerate) GS is a simple product of a completely filled (A or B) chain by a completely empty (B or A) one. Note that the symmetrized GS (i.e. the equal weight superposition of the two classical GS) still retains a $1/L$ finite size entropy as shown in Fig. 2(a). Secondly, each quantum phase is uniquely characterized by its ES defined as the spectrum of $\mathcal{H} = -\ln \rho_A$: Fig. 2(b-c) (Fig. 2(d-e)) for parameters in the CDW phase (LL phase) shows very distinctive features and a clear gapped (linear gapless) spectrum. In addition, at the QCP (Fig. 2(d)) one observes a SU(2) multiplet structure.

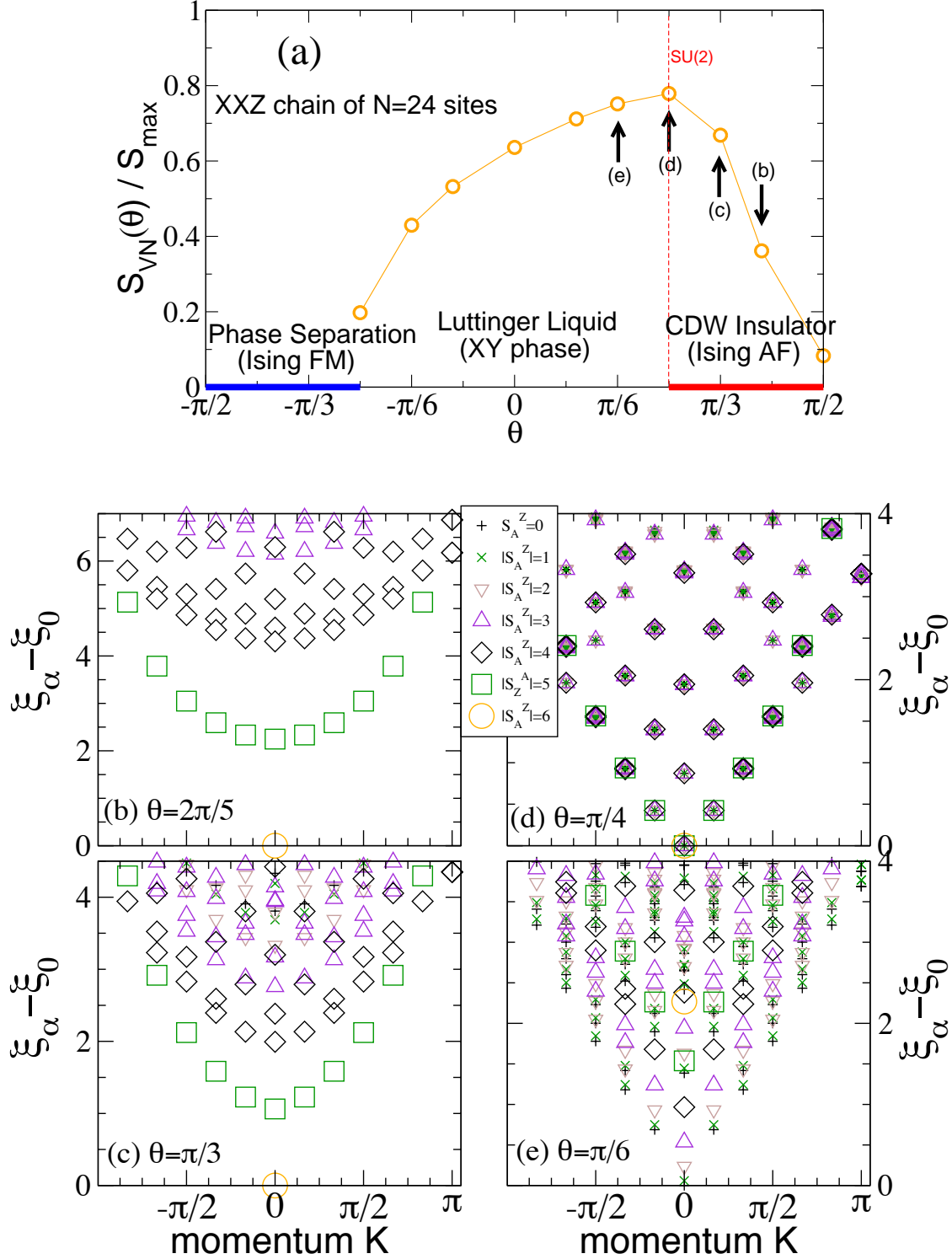


Figure 2. (Color online) GS properties of the XXZ chain; (a) VN entanglement entropy vs θ computed on a $N=24$ site ring. The entropy is normalized by the maximum value $S_{\max} = L \ln 2$ ($L = N/2$). (b-e) Typical entanglement *excitation* spectra (for the 4 values of θ shown in (b)) as a function of momentum K along the ribbon. The eigenvalues ξ_α of $\mathcal{H} = -\ln \rho_A$ are labelled according to the Z-component of the total spin (i.e. the number of bosons $N_A = L/2 - S_A^Z$) of the A subsystem (legend of symbols on graph). Note the CDW (LL), doubly-degenerate (unique) GS of \mathcal{H} (of energy ξ_0) carries $N_A = 0$ or $N_A = L$ ($N_A = L/2$) hardcore bosons.

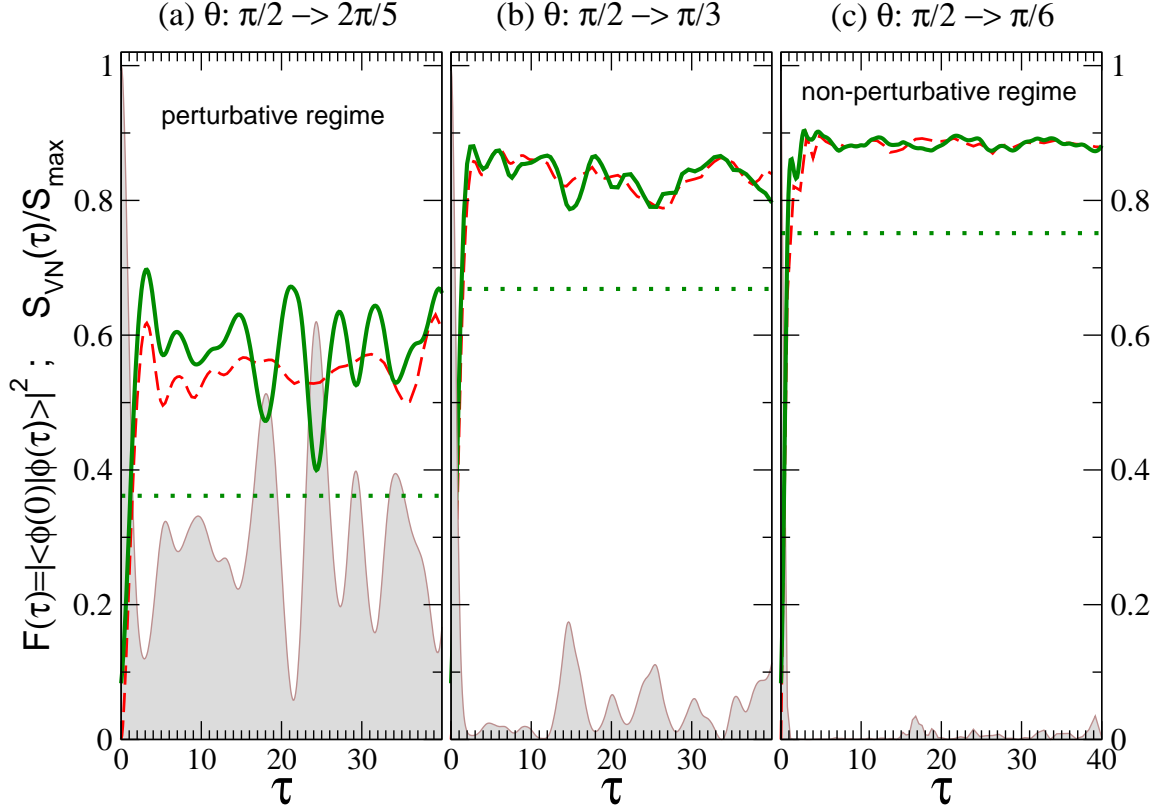


Figure 3. (Color online) (a-c) Squared-fidelity (shaded) and VN entanglement entropy of various non-equilibrium states obtained after different *sudden* quenches of the $N = 24$ sites chain Hamiltonian, $\theta_{\text{init}} \rightarrow \theta_f$ as shown on plots ($t' = V' = 0$). The continuous lines (dashed red line) correspond to a symmetric (non-symmetric) initial state (see text). The dotted (green) line is the GS entropy for $\theta = \theta_f$.

3. Time evolution and bipartite entanglement entropy

Let us now consider a *sudden* quantum quench of the system at time $\tau = 0$ (quasi-adiabatic quenches will be treated later on). For simplicity, the initial state $|\phi(0)\rangle$ is chosen either (i) as one of the two (zero entropy) degenerate GS at $\theta = \theta_{\text{init}} = \pi/2$ (i.e. for vanishing hopping) shown in Fig. 1(c), where all hardcore bosons are located on a single edge, A or B, or (ii) as a symmetric combination of the two. At times $\tau > 0$, the boson hopping t (spin-flip term in spin language) is switched on, i.e. the value of θ discontinuously jumps at $\tau = 0$ to its "final" value θ_f (see Fig. 1(b)).

The time evolution of the system wavefunction,

$$|\phi(\tau)\rangle = \exp(-i\tau H(\theta_f))|\phi(0)\rangle \quad (2)$$

is easily computed by (arbitrary) time steps of $\delta\tau$ (from 0.01 to 0.8) with arbitrary good precision [22] (typically better than 10^{-16}) by Taylor expanding the time evolution

operator $\exp(-i\delta\tau H(\theta_f))$. Hence, for finite size systems under consideration, an *exact* computation of the time-dependent RDM,

$$\rho_A(\tau) = \text{Tr}_B |\phi(\tau)\rangle\langle\phi(\tau)| \quad (3)$$

can be done. Results for the squared-fidelity $F(\tau) = |\langle\phi(0)|\phi(\tau)\rangle|^2$ and the entanglement entropy,

$$S_{\text{VN}}(\tau) = -\text{Tr}\{\rho_A(\tau) \ln \rho_A(\tau)\} \quad (4)$$

are shown in Fig. 3 for increasingly "strong" quenches corresponding to $\theta_f = 2\pi/5$, $\pi/3$ and $\pi/6$. After a very short transient regime the squared-fidelity drops sharply and the entanglement entropy raises to a more or less well defined plateau, whose average value exceeds the value of the GS of the final Hamiltonian (f-GS). In the $N \rightarrow \infty$ limit, one expects a non-perturbative regime [23] where $F(\tau) \sim 2^{-N} \rightarrow 0$. However, on finite system and for small quench, $F(\tau)$ can remain large, as seen e.g. in Fig. 3(a). In that regime, integrability of the model can play an important role [24]. Therefore, from now on, one will assume a sufficiently large quench, let say $\theta_f = \pi/6$, to observe time evolutions on finite clusters *generic of the thermodynamic limit*. This corresponds in fact to a quench "across" the QCP at $\theta = \pi/4$ into the region of stability of the LL phase. However, from Fig. 3(c) we can see a well defined entropy plateau, suggesting that the system does reach a steady state after the quench. Note also that one finds that non-symmetric and symmetrized initial states give very similar results so that, for simplicity, one can restrict to the symmetric case hereafter.

The time evolution of any (static) observable O of the A subsystem is given by $\tilde{O}(\tau) = \text{Tr}(\rho_A(\tau)O)$. Time-average like $\bar{O} = \langle\tilde{O}\rangle$, where $\langle G \rangle \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T G(\tau) d\tau$, can then be rewritten as $\text{Tr}(\rho_A^{\text{ave}} O)$, involving the *time-averaged* RDM $\rho_A^{\text{ave}} = \langle\rho_A\rangle$. Next, time-averages will be performed in a $\Delta\tau = 40$ time interval using a mesh of 50 points (excluding the initial short transient regime).

Bipartite entanglement can provide precise characterization of the system after the quench, in the thermodynamic limit, e.g. showing whether or not it reaches a (quasi-) steady state. One finds that the time fluctuations of $S_{\text{VN}}(\tau)$ shown in Fig. 4(a) vanish in the thermodynamic limit as revealed by the finite size scaling of Fig. 4(c). Interestingly, as seen in Fig. 4(a), we notice that the VN entropy of the average density matrix ρ_A^{ave} , $S_{\text{VN}}^{\text{ave}} = -\text{Tr}\{\rho_A^{\text{ave}} \ln \rho_A^{\text{ave}}\}$, differs from $\langle S_{\text{VN}} \rangle$, the time-average of the entanglement entropy. A proper finite size scaling of the two quantities shown in Fig. 4(b) proves that this fundamental property holds in the thermodynamic limit.

Incidentally, it is also important to distinguish between "quantum" fluctuations $\overline{(O - \bar{O})^2} = \text{Tr}\{\rho_A^{\text{ave}}(O - \bar{O})^2\}$ and "time" fluctuations $\langle(\tilde{O} - \bar{O})^2\rangle$. An extreme example is the case of S_A^Z where $\tilde{S}_A^Z(\tau) = 0$ at all times (from the time-conserved $A \leftrightarrow B$ symmetry) while quantum fluctuations are finite as shown in Fig. 4(c).

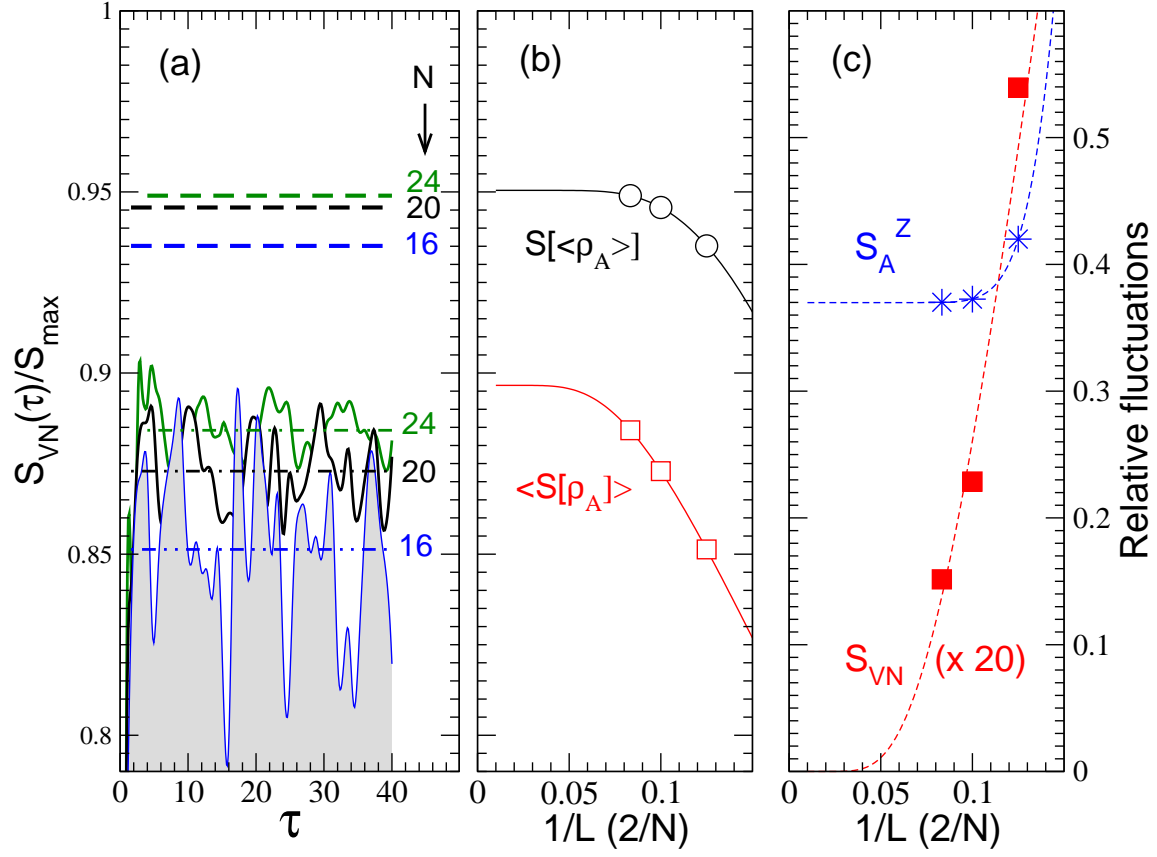


Figure 4. (Color online) (a) Time-dependent VN entanglement entropy after a *sudden* $\theta_{\text{init}} = \pi/2 \rightarrow \theta_f = \pi/6$ quench ($t' = V' = 0$), for different chain lengths N as shown on plots. Dashed-dotted (dashed) lines corresponds to $\langle S_{VN} \rangle$ (S_{VN}^{ave}). (b) Finite-size scaling of the averages shown in (a). (c) Finite-size scaling of the relative time fluctuations of the entanglement entropy (red squares) and quantum fluctuations of S_A^Z (normalized by $L/2$). Time-averages are all performed in a $\Delta\tau = 40$ time interval using a mesh of 50 points. The data of b) and c) are well fitted assuming $\sim 2^{-\alpha L}$ finite size corrections.

4. Entanglement spectra: time evolution within the "entropy plateau"

In order to understand the physical origin of the difference between $\langle S_{VN} \rangle$ and S_{VN}^{ave} it is interesting to inspect more closely the behavior of $\rho_A(\tau)$ within the "entropy plateau" regime as a function of time τ . For this purpose, it is convenient to use the (bipartite) *time-dependent* ES defined by the (positive) eigenvalues of the (dimensionless) pseudo-Hamiltonian $\mathcal{H}(\tau)$ defined as,

$$\rho_A(\tau) = \exp(-\mathcal{H}(\tau)). \quad (5)$$

As for the equilibrium GS, the spectrum of \mathcal{H} is computed separately in every sector of the momentum $K = \frac{2\pi}{L/2}$ (using translation invariance of the A and B half-systems) and of the z-component of the total spin. Fig. 5 shows "typical" entanglement spectra taken

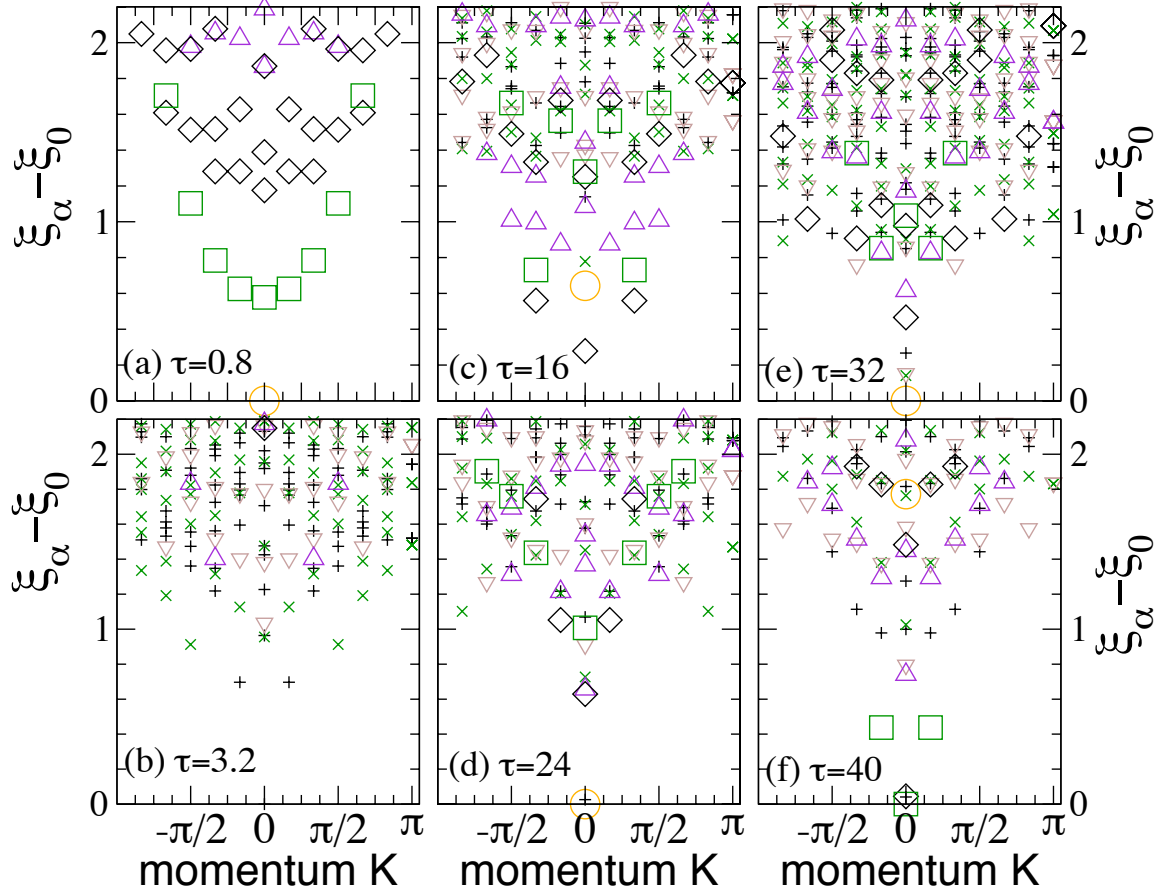


Figure 5. (Color online) "Snapshots" at different times τ of the ES of ρ_A . (a) lies within the transient regime and (b-f) lie within the "entropy plateau". Symbols are similar to Fig. 2(b-e).

at different times. At short times, some memory of the initial state is clearly visible as in Fig. 5(a). At longer times, let's say $\tau > 2.5$, it is remarkable to find very different spectra at different times although all such spectra lead roughly to a very comparable value of the entanglement entropy. In other words, $\rho_A(\tau)$ fluctuates strongly in time on a *constant entropy* "hyper-surface" of the space of density matrices with $\text{Tr} \rho = 1$. This naturally explains why $\langle S_{\text{VN}} \rangle$ and $S_{\text{VN}}^{\text{ave}}$ differ substantially. Our finite size scaling suggests that this is a fundamental phenomenon and not a finite size effect.

5. Entanglement spectra: comparison between infinite-time average and thermal ensembles

The RDM ρ_A contains all relevant information about the subsystem, much beyond any local observable. As argued in the introduction, its associated ES is therefore an ultimate observable to investigate thermalization. Furthermore, the existence of *conserved quantities* such as momentum and particle number makes the comparison

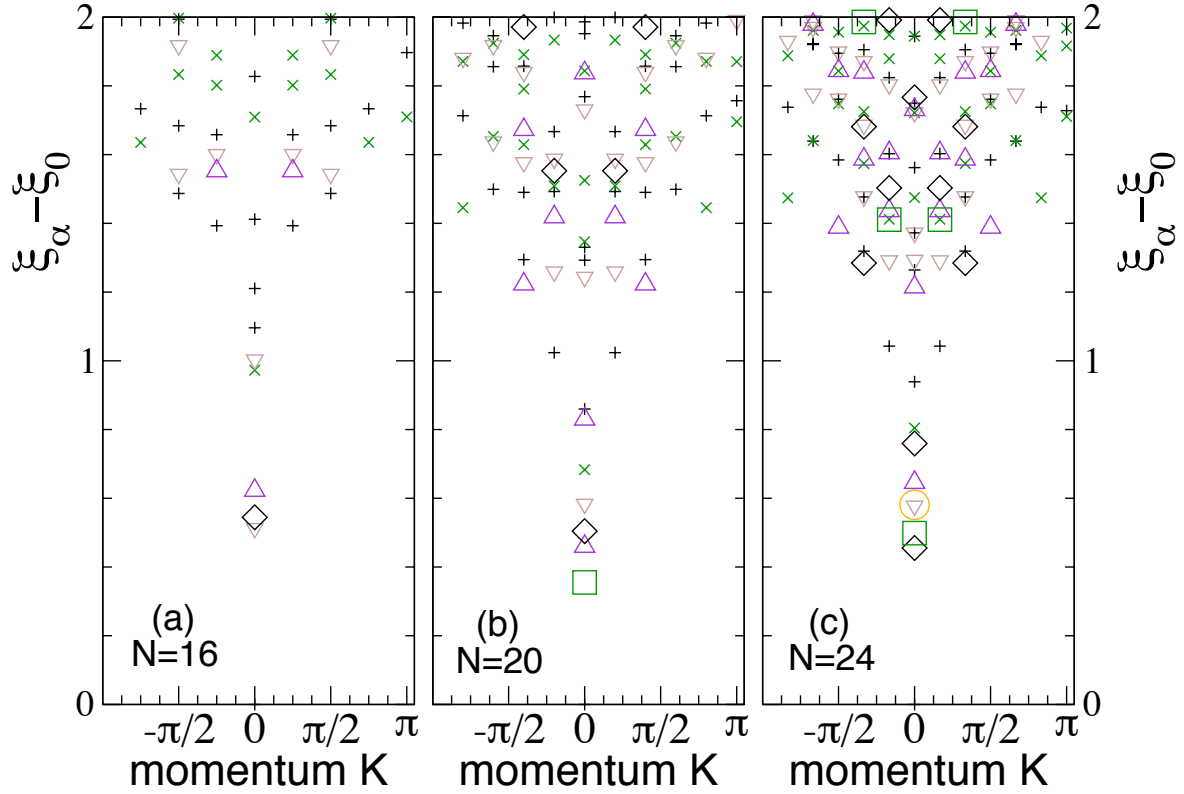


Figure 6. (Color online) Entanglement spectra of the *time-averaged* RDM obtained after a $\theta_{\text{init}} = \pi/2 \rightarrow \theta_f = \pi/6$ quench in XXZ chains of different lengths, $N = 16, 20$ and 24 ($t' = V' = 0$). Symbols are similar to Fig. 2(b-e) and time-averages are performed as for Fig. 4.

between time-average and thermal density matrices much finer, providing a stringent thermalization criterion (based on the close similarity between these two quantities).

The ES of ρ_A^{ave} computed for XXZ chains (in the entropy plateau) with 16, 20 and 24 sites, and shown in Fig. 6(a-c), reveal a smooth convergence with system size, an accumulation of low pseudo-energy levels at $K = 0$ and a small pseudo-energy gap, in sharp contrast with the ES of the f-GS shown in Fig. 2(e).

In order to probe thermalization, we now wish to compare ρ_A^{ave} to the thermal average of ρ_A . Thermal averages can be defined as:

$$\rho_A^\lambda = \sum_\alpha^I w_\alpha^\lambda \text{Tr}_B |\Psi_\alpha\rangle \langle \Psi_\alpha|, \quad (6)$$

where the prime means the sum is restricted to eigenstates of $H(\theta_f)$ with *non-zero* overlap with $|\phi(0)\rangle$, accounting for conserved quantities like momentum and z-component of total spin, and hidden conservation laws in the integrable case. Both canonical ($w_\alpha^{\text{can}} = \frac{1}{Z} \exp(-\beta E_\alpha)$ where E_α are eigenenergies associated to $|\Psi_\alpha\rangle$) and

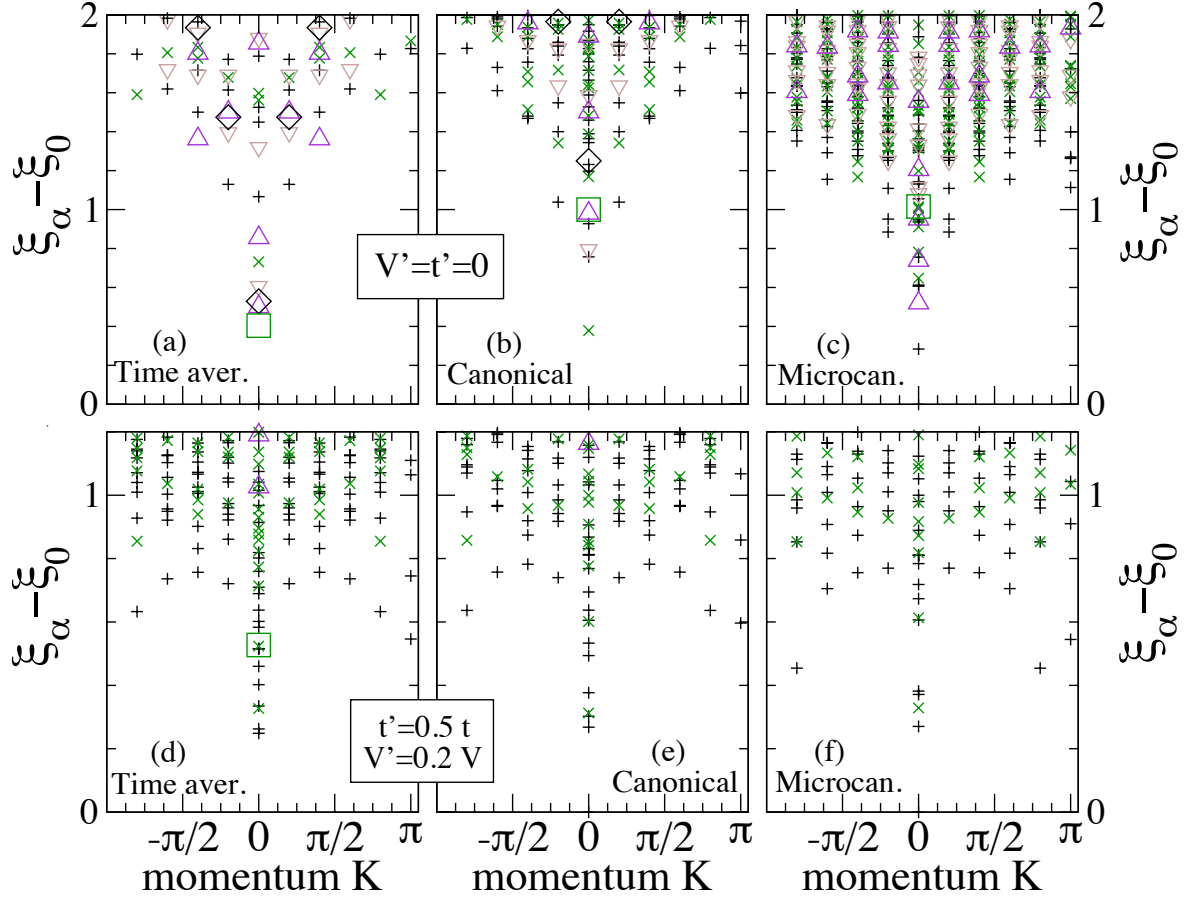


Figure 7. (Color online) ES of ρ_A^{ave} (a,d), ρ_A^{can} (b,e) and ρ_A^{micro} (c,f) obtained by full diagonalization of $N = 20$ site integrable ($t' = V' = 0$) and non-integrable chains ($\theta_{\text{init}} = \pi/2 \rightarrow \theta_f = \pi/6$ quench). Symbols are similar to Fig. 2(b-e). $1/\beta \simeq 2.07$ (b), $\Delta E = 0.3$ (c), $1/\beta \simeq 1.48$ (e) and $\Delta E = 0.1$ (f).

microcanonical ($w_\alpha^{\text{micro}} = \text{Cst}$ in an energy window $[E_0 - \Delta E, E_0 + \Delta E]$) thermal ensembles can be considered. Note, the effective temperature $1/\beta$ of the canonical ensemble is (implicitly) given by constraining the conserved mean-energy to be $E_0 = -(V - V')L/2$. Note that a full diagonalization of $H(\theta_f)$ is now required since the previous formula (6) involves the complete set of eigenstates $|\Psi_\alpha\rangle$, hence limiting the available chain lengths to $N = 20$. The *infinite-time* average ρ_A^{ave} can be computed exactly also using (6) with $w_\alpha^{\text{ave}} = |\langle \Psi_\alpha | \phi(0) \rangle|^2$ which, in contrast to the thermal averages, depends now on the initial state. Incidentally, this enables to control the very good accuracy of the previous approximate averaging procedure, as shown by a direct comparison between Fig. 6(b) and Fig. 7(a).

The results shown in Figs. 7 reveal striking differences between integrable and non-integrable Hamiltonians. For the latter, surprisingly good agreement between the low-energy ES of ρ_A^{ave} and ρ_A^{can} is seen, suggesting that the *Eigenstate Thermalization Hypothesis* [10] may apply up to the level of the (bipartite) entanglement properties of

eigenstates. In contrast, for the integrable XXZ chain, noticeable differences between the various averages reveal incomplete thermalization which seems to be a fundamental property of such systems.

6. Quasi-adiabatic quenches

The system after the quench possesses an excess of entanglement entropy (per unit length) compared to the f-GS. There is in fact an interesting correlation between this excess entropy and the "speed" at which the quench is performed. A smooth quench can be realized by considering a time-dependent Hamiltonian $H(\theta(\tau))$ where $\theta(\tau)$ decreases continuously from $\theta_{\text{init}} = \pi/2$ to $\theta_f = \pi/6$ in a time-interval $[0, T_f]$. One can choose e.g.

$$\theta(\tau) = (1 - \tilde{w}_\tau)\theta_{\text{init}} + \tilde{w}_\tau\theta_f, \quad (7)$$

$$\tilde{w}_\tau = (w_\tau - w_0)/(w_{T_f} - w_0), \quad (8)$$

$$w_\tau = 1/(1 + \exp((T_f/2 - \tau)/T_1)), \quad (9)$$

and $T_1 = T_f/10$. As shown in Fig. 8, under increasing the characteristic time T_1 the average value of the entropy plateau decreases towards the GS value, as expected for increasingly adiabatic processes. Interestingly, the deviation $\langle S_{\text{VN}} \rangle - S_{\text{VN}}^{\text{ave}}$ also simultaneously decreases (to reach zero for a fully adiabatic process).

Sudden and continuous quenches also give very different ES of the time-averaged reduced density matrix ρ_A^{ave} . Data for continuous and sudden quenches are compared in Fig. 9. Remarkably, the spectra in the quasi-adiabatic case resemble very closely the equilibrium f-GS spectrum of Fig. 2(e) of the main paper with a clear linear envelope, in contrast to the spectra obtained for a sudden quench. However, we note that the requirement (e.g. on the time scale T_1) to have a quasi-adiabatic process becomes more and more stringent for increasing system size because of the vanishing of the finite-size gap (in the XY phase).

7. Summary

To summarize, the concept of bipartite entanglement is introduced in a genuine out-of-equilibrium *isolated* many-body system. I propose that it provides a simple and complete probe of thermalization, beyond the investigation of simple local observables. In this framework, absence of thermalization is diagnosed whenever the reduced density matrix deviates (once averaged over a sufficiently large time interval) noticeably from the thermal (reduced) density matrix taken at an effective temperature imposed only by the initial conditions. Generically, we find that the system approaches quickly an entanglement entropy plateau. However, integrable and non-integrable chains behave quite differently within this plateau. The RDM of the integrable chain strongly fluctuates in time and does not thermalize according to the above criterium. This is to be contrasted to the case where extra terms are introduced in the chain Hamiltonian to break the integrability: in such a case, the ES of the RDM and the spectrum of

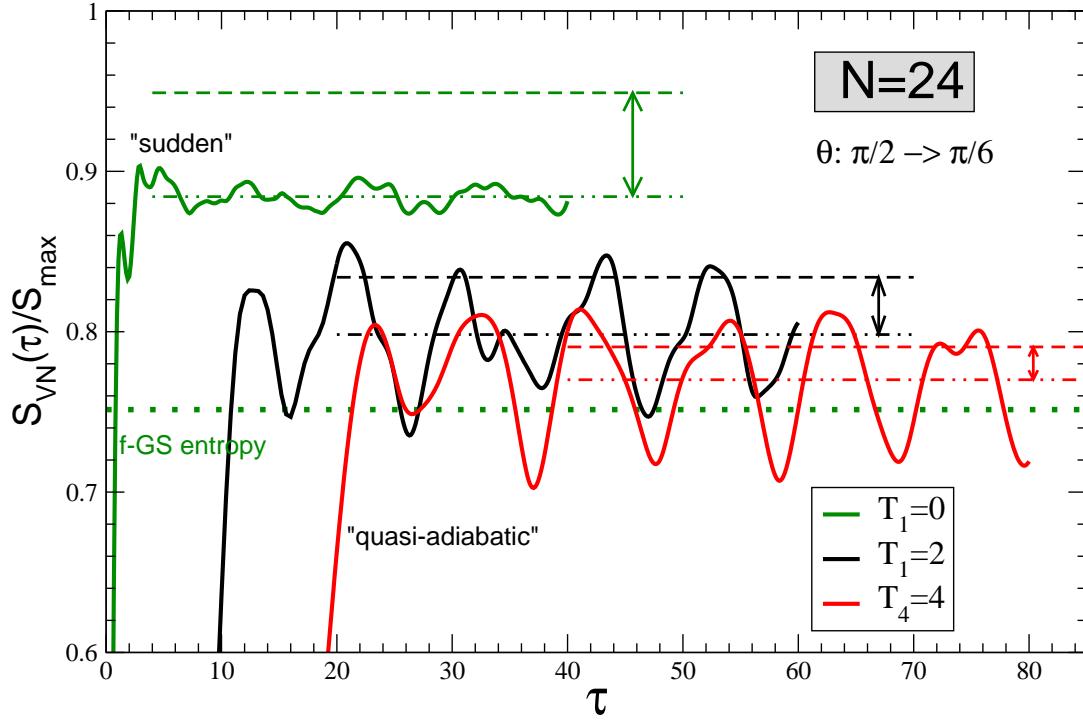


Figure 8. (Color online) VN entanglement entropy of various non-equilibrium states obtained after a $\theta_{\text{init}} = \pi/2 \rightarrow \theta_f = \pi/6$ quench realized on a 24-site XXZ chain ($t' = V' = 0$) over different time scales characterized by T_1 . The arrows indicate the difference between the time-averaged entropy $\langle S_{\text{VN}} \rangle$ (dot-dashed lines) and the VN entropy of ρ_A^{ave} , $S_{\text{VN}}^{\text{ave}}$ (dashed lines). Time-averages have been performed using 50 bins in a $\Delta\tau = 40$ time-interval within the plateaux.

the thermal density matrix become very similar, a strong hint that thermalization can occur up to the level of an extensive subsystem even though the full system is completely isolated.

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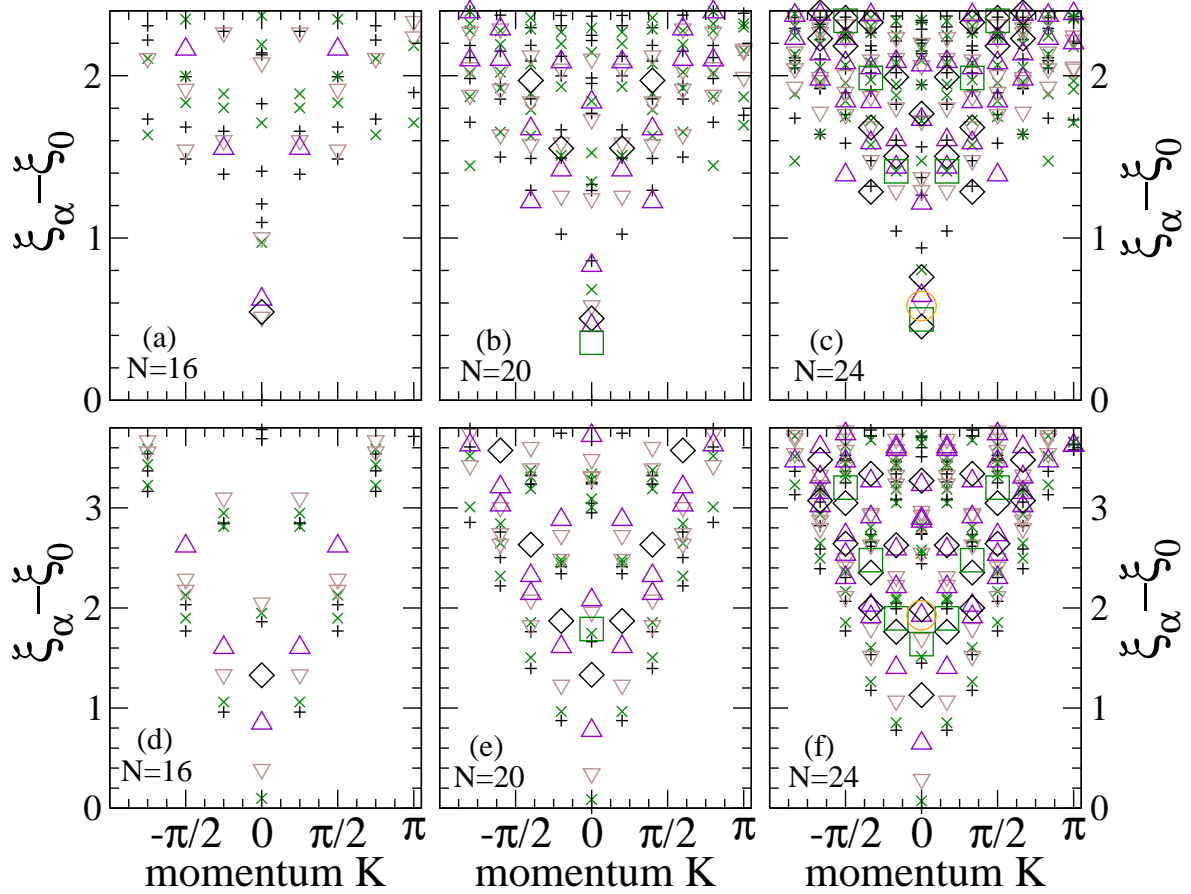


Figure 9. (Color online) Entanglement spectra of *time-averaged* reduced density matrices in XXZ chains of different lengths, $N = 16, 20$ and 24 ($t' = V' = 0$). Comparison between a sudden quench (a-c) (data similar to Fig. 6 reproduced here for convenience) and a quasi-adiabatic $T_1 = 4$ quench (d-f). In both cases, $\theta_{\text{init}} = \pi/2 \rightarrow \theta_f = \pi/6$.

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